

COMPOSITION SERIES FOR DEGENERATE PRINCIPAL SERIES OF $GL(n)$

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ABSTRACT. In this note we consider representations of the group $GL(n, F)$, where F is the field of real or complex numbers or, more generally, an arbitrary local field, in the space of equivariant line bundles over Grassmannians over the same field F . We study reducibility and composition series of such representations.

Similar results were obtained already in [HL99, Al12, Zel80], but we give a short uniform proof in the general case, using the tools from [AGS15a]. We also indicate some applications to cosine transforms in integral geometry.

1. INTRODUCTION

Let F be a local field, *i.e.* either \mathbb{R} or \mathbb{C} or a finite extension of the field \mathbb{Q}_p of p -adic numbers for some prime p , or the field $\mathbb{F}_q((t))$ of Laurent series over a finite field. Let $G = G_n := GL(n, F)$ be the group of invertible matrices of size n with entries in F .

Then $C^\infty(G)$ is a $G \times G$ -module with left and right actions given by $L_g f(x) = f(g^{-1}x)$, $R_g f(x) = f(xg)$. Let $P \subset G$ be a Lie subgroup with modular function Δ_P and let χ be a character of P . The induced representation $I(P, \chi) := \text{Ind}_P^G(\chi)$ is the right G -action on the space

$$(1) \quad C^\infty(G, P, \chi) := \left\{ f \in C^\infty(G) \mid L_{p^{-1}} f = \chi(p) \Delta_P^{1/2}(p) f \text{ for all } p \in P \right\},$$

whose elements may also be regarded as smooth sections of a line bundle on G/P .

We are interested in the so-called maximal parabolic subgroups $P = P_{p_1, p_2}$, with $p_1 + p_2 = n$, consisting of matrices $x \in G_n$ of the form

$$(2) \quad x = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}; x_{ij} \in \text{Mat}_{p_i \times p_j}.$$

In this case G/P is the Grassmannian of p_1 -dimensional subspaces of F^n . The characters of P are of the form $\chi_1 \otimes \chi_2(x) = \chi_1(x_{11}) \chi_2(x_{22})$, where χ_i is a character of G_{p_i} . Following [BZ77] we write $\chi_1 \times \chi_2$ instead of $I(P, \chi_1 \otimes \chi_2)$.

It is well known that the representation $\chi_1 \times \chi_2$ is irreducible for most characters χ_1, χ_2 . We explicitly describe the exceptional characters for which $\chi_1 \times \chi_2$ is reducible, and show that the composition series are monotone in a certain sense. We also show that if F is non-archimedean, the length of $\chi_1 \times \chi_2$ is at most two, *i.e.* it has at most one non-trivial G -invariant subspace.

In 1.1 below we describe the main results precisely. In 1.2 below we give applications to integral geometry.

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Most of the results are known, see 1.3. However, we give short proofs that work uniformly for all fields. Our main tool are the Bernstein-Zelevinsky derivatives ([BZ77]) and their archimedean analogs developed in [AGS15a, AGS15b]. We describe these, as well as other preliminaries, in §2.

1.1. Main results. We fix some notation to describe our main results succinctly. For $z \in F$ let $\nu(z)$ denote the positive scalar by which the additive Haar measure on F transforms under multiplication by z . We will also regard ν as a character of G_k defined by $\nu(g) := \nu_k(g) := \nu(\det g)$, and we note that the modular function of $P = P_{p_1, p_2}$ is $\Delta_P = \nu^{p_2} \otimes \nu^{-p_1}$. We also let 1 denote the trivial character of G_k , ε denote the $\text{sgn}(\det)$ character of $\text{GL}_k(\mathbb{R})$, and α denote the character $\det|\det|^{-1}$ of $\text{GL}_k(\mathbb{C})$. Twisting by a character of G_n we can assume $\chi_2 = 1$ and consider $\chi \times 1$. Let $r := \min(p_1, p_2)$.

Theorem 1.1. *$\chi \times 1$ is reducible if and only if one of the following holds*

- (i) $\chi = \nu^{\pm(k-n/2)}$, where k is an integer, and $0 \leq k \leq r-1$.
- (ii) $F = \mathbb{R}$, and $\chi = \varepsilon^{k+1} \nu^{\pm(k-r+n/2)}$ for some positive integer k .
- (iii) $F = \mathbb{R}$, $r > 1$ and $\chi = \nu^{\pm(k-r+n/2+1)}$ or $\chi = \varepsilon \nu^{\pm(k-r+n/2+1)}$ for some positive integer k .
- (iv) $F = \mathbb{C}$, and either χ or χ^{-1} equals $\alpha^k \nu^s$ where $s - n/2$ and k are integers and $s + k, s - k > n/2 - r$.

We also show (Lemma 4.2) that if F is non-archimedean then $\text{length}(\chi \times 1) \leq 2$.

Now we want to show that $\chi \times 1$ has a unique composition series, monotone in some sense. For Archimedean F we could use just the Gelfand - Kirillov dimension. For the general case we will use the notion of rank, see §2.4. Define $s(\chi) \in \mathbb{R}$ by $|\chi| = \nu^{s(\chi)}$.

Theorem 1.2. *If $s(\chi) > 0$ then the irreducible constituents of $\chi \times 1$ appear in strictly descending order of rank. Otherwise they appear in strictly ascending order of rank.*

Consider the standard intertwining operator $I_\chi : \chi \times 1 \rightarrow 1 \times \chi$. From Theorem 1.2 we obtain

Corollary 1.3. *The image of I_χ is the only irreducible quotient of $\chi \times 1$ and the only irreducible submodule of $1 \times \chi$. In particular, I_χ is invertible if and only if $\chi \times 1$ is irreducible.*

1.2. Connections to integral geometry. The α -cosine transform on real Grassmannian in a Euclidean space \mathbb{R}^n , including the case of even functions on the sphere, was studied by both analysts and geometers for a long period of time: [Sem61, Sch70, Kol97, OR05, Rub98, Rub02, Rub13]. The case $\alpha = 1$ plays a special role in convex and stochastic geometry, see [GH90a, GH90b, GH96], and in particular in valuation theory [AB04, Al03, Al04, Al11]. One of the reasons why the case of at least some values $\alpha \neq 1$ is relevant for geometry is the recent result [AlGS15] saying that the Radon transform between the Grassmannians of i - and $(n-i)$ -dimensional linear subspaces in \mathbb{R}^n coincides with the $\alpha = -\min\{i, n-i\}$ -cosine transform¹, and moreover sometimes composition of two Radon transforms coincides with the α -cosine transform for appropriate $\alpha \neq 1$.

¹For $i = 1$ this result is known for a long time and seems to be a folklore.

Let us recall the definition of the α -cosine transform. Let $Gr_i(\mathbb{R}^n)$ denote the Grassmannian of i -dimensional linear subspaces in the Euclidean space \mathbb{R}^n . For a pair of subspaces $E, F \in Gr_i(\mathbb{R}^n)$ one defines (the absolute value of) the cosine of the angle $|\cos(E, F)|$ between E and F as the coefficient of the distortion of measure under the orthogonal projection from E to F . More precisely, let $q: E \rightarrow F$ denote the restriction to E of the orthogonal projection $\mathbb{R}^n \rightarrow F$. Let $A \subset E$ be an arbitrary subset of finite positive Lebesgue measure. Then define

$$|\cos(E, F)| := \frac{vol_F(q(A))}{vol_E(A)},$$

where vol_F, vol_E are Lebesgue measures on F, E respectively induced by the Euclidean metric on \mathbb{R}^n normalized so that the Lebesgue measures of unit cubes are equal to 1. It is easy to see that $|\cos(E, F)|$ is independent of the set A and is symmetric with respect to E and F .

Example 1.4. *If $i = 1$ then $|\cos(E, F)|$ is the usual cosine of the angle between two lines.*

For $\alpha \in \mathbb{C}$ with $Re(\alpha) \geq 0$ let us define the α -cosine transform

$$T_\alpha := T_\alpha^i: C^\infty(Gr_i(\mathbb{R}^n)) \rightarrow C^\infty(Gr_i(\mathbb{R}^n))$$

by

$$(3) \quad (T_\alpha^i f)(E) = \int_{F \in Gr_i(\mathbb{R}^n)} |\cos(E, F)|^\alpha f(F) dF,$$

where dF is the $O(n)$ -invariant Haar probability measure on $Gr_i(\mathbb{R}^n)$.

Remark 1.5. *One can show that the integral (3) absolutely converges for $Re(\alpha) > -1$ (see [Al12, Lemma 2.1]).*

It is well known that T_α has a meromorphic continuation in $\alpha \in \mathbb{C}$. For $\alpha_0 \in \mathbb{C}$ we will denote by S_{α_0} the first non-zero coefficient in the decomposition of the meromorphic function T_α near α_0 , namely

$$T_\alpha = (\alpha - \alpha_0)^k \cdot (S_{\alpha_0} + O(\alpha - \alpha_0)) \text{ as } \alpha \rightarrow \alpha_0, S_{\alpha_0} \neq 0.$$

Thus

$$S_{\alpha_0}: C^\infty(Gr_i(V)) \rightarrow C^\infty(Gr_i(V))$$

is defined and non-zero for any $\alpha_0 \in \mathbb{C}$, and coincides with T_{α_0} for all but countably many α_0 (necessarily $Re(\alpha_0) \leq -1$ for exceptional values of α_0). S_α will also be called the α -cosine transform and sometimes denoted S_α^i . The geometric meaning of S_{α_0} for the exceptional values of α_0 is investigated in [AlGS15].

Obviously the α -cosine transform commutes with the natural action of the orthogonal group $O(n)$. It was observed in [AB04] for $\alpha = 1$ and in [Al12] for general α that the α -cosine transform can be rewritten in such a way to commute with an action of the much larger full linear group $GL(n, \mathbb{R})$. In that language it is a map

$$(4) \quad C^\infty(Gr_i(\mathbb{R}^n), L_\alpha) \rightarrow C^\infty(Gr_{n-i}(\mathbb{R}^n), M_\alpha),$$

where L_α, M_α are appropriately chosen $GL(n, \mathbb{R})$ -equivariant line bundles over Grassmannians, and a choice of a Euclidean metric on \mathbb{R}^n induces identification $Gr_{n-i}(\mathbb{R}^n) \simeq Gr_i(\mathbb{R}^n)$ by taking the orthogonal complement, and $O(n)$ -equivariant identifications of L_α and M_α with trivial line bundles. In fact the α -cosine transform turned out to be a special case of the standard representation theoretical construction called

intertwining integral. This opened the way to apply the infinite dimensional representation theory of the group $GL(n, \mathbb{R})$. In that language we have

$C^\infty(Gr_i(\mathbb{R}^n), L_\alpha) \simeq \nu_i^{(n-i)/2} \times \nu_{n-i}^{-\alpha-i/2}$, $C^\infty(Gr_{n-i}(\mathbb{R}^n), M_\alpha) \simeq \nu_{n-i}^{-\alpha-i/2} \times \nu_i^{(n-i)/2}$, and S_α is the standard intertwining operator

$$S_\alpha^i : \nu_i^{(n-i)/2} \times \nu_{n-i}^{-\alpha-i/2} \rightarrow \nu_{n-i}^{-\alpha-i/2} \times \nu_i^{(n-i)/2}.$$

Thus Theorem 1.1 and Corollary 1.3 immediately imply the following corollary.

Corollary 1.6. (1) S_α^i is not invertible if and only if there exists a non-negative integer k such that one of the following holds: (a) $\alpha = 2k$, (b) $\alpha = -n - 2k$, (c) $r > 1$ and $\alpha = 1 - r + k$, (d) $r > 1$ and $\alpha = r - n - 1 - k$.
 (2) If S_α^i is not invertible then its range is the intersection of all closed G -invariant proper subspaces of $\nu_{n-i}^{-\alpha-i/2} \times \nu_i^{(n-i)/2}$.
 (3) If S_α^i is invertible then its inverse is a scalar multiple of $S_{-n-\alpha}^{n-i}$.

1.3. Related works. In the archimedean cases Theorem 1.1 was proven earlier in [HL99], the non-archimedean case follows from [Zel80]. In fact [HL99] computes the length of $\chi_1 \times \chi_2$ in the archimedean cases. For $F = \mathbb{R}$ and $\chi = \nu^s$, where $|s| + n/2$ is not an integer, or is bigger than $n - 1$ Theorem 1.2 was proven in [Al12]. For $F \in \{\mathbb{R}, \mathbb{C}\}$ and $p_1 \geq p_2$ a similar theorem was proven in [HL99], where the irreducible constituents are characterized by their K -types, as opposed to the rank (or Gelfand-Kirillov dimension). The assumption $p_1 \geq p_2$ can be removed using the Cartan involution $g \mapsto {}^t g^{-1}$. For $F = \mathbb{R}$, $n = 2r$ this theorem was proven in [BSS90].

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2. PRELIMINARIES

In this section we recall some of the necessary background for the benefit of the reader.

2.1. Degenerate principal series. Let G be a reductive algebraic group over an arbitrary local field F . In this section we discuss some basic properties of the induced representation $I(P, \chi)$ on $C^\infty(G, P, \chi)$ as in (1). For detailed proofs we refer the reader to [BW80, Wall88] or to other standard texts on representation theory.

Let $\mathcal{E}'(G)$ denote the set of compactly supported distributions on G , regarded as a left and right G -module as usual via the pairing $\langle \cdot, \cdot \rangle : \mathcal{E}'(G) \times C^\infty(G) \rightarrow \mathbb{C}$.

Lemma 2.1. Let $\varepsilon \in \mathcal{E}'(G)$ denote evaluation at $1 \in G$, then we have

$$\langle R_{p^{-1}} \varepsilon, f \rangle = \chi(p) \Delta_P^{1/2}(p) \langle \varepsilon, f \rangle \text{ for all } p \in P, f \in C^\infty(G, P, \chi)$$

Proof. Indeed both sides are equal to $f(p)$. □

Lemma 2.2 ([Wall88, V.5.2.4]). The representations $I(P, \chi)$ and $I(P, \chi^{-1})$ are contragredient.

Let \bar{P} denote the parabolic subgroup opposite to P . Then the characters of P and \bar{P} can be identified with those of the common Levi subgroup $L = P \cap \bar{P}$.

Proposition 2.3 ([KnSt71, Wald03]). *There is a nonzero intertwining operator $I(P, \chi) \rightarrow I(\bar{P}, \chi)$.*

Lemma 2.4. *If K is a closed Lie subgroup (or an l -subgroup) of G such that $PK = G$ then the restriction $I(P, \pi)|_K$ is isomorphic to the induced representation $Ind_{P \cap K}^K(\pi|_{P \cap K})$. In particular, the space $I(P, \pi)^K$ of K -fixed vectors is isomorphic to $\pi^{P \cap K}$.*

The isomorphism is given just by restricting the functions to K . We will use this lemma for the maximal compact subgroup $K \subset G$ which in the non-Archimedean case is (up to conjugation) the subgroup $GL(n, \mathcal{O})$, where \mathcal{O} is the ring of integers of F . For $F = \mathbb{R}$, K is the orthogonal group $O(n)$ and for $F = \mathbb{C}$, $K = U(n)$.

In both Archimedean cases, $P \cap K$ is a symmetric subgroup of K and we will use the following well-known lemma.

Lemma 2.5 (see e.g. [Wol07, Theorem 12.2.4]). *If G is a connected and compact Lie group, M is a symmetric subgroup of G and χ is a character of M then $Ind_M^G(\chi)$ is a multiplicity-free representation.*

2.2. Finite dimensional representations. Let (ϕ, V) be an irreducible finite dimensional representation of a reductive group G . We are interested in the possibility of realizing ϕ as a submodule or quotient of some $I(P, \chi)$, which we denote by $\phi \hookrightarrow I(P, \chi)$ and $I(P, \chi) \twoheadrightarrow \phi$ respectively. Let (ϕ^*, V^*) be the contragredient representation of (ϕ, V) .

From Lemmas 2.1 and 2.2 we obtain

Lemma 2.6. *We have $\phi \hookrightarrow I(P, \chi)$ (resp. $I(P, \chi) \twoheadrightarrow \phi$) if and only if $(V^*)^{P, \chi^{-1} \Delta_P^{-1/2}}$ (resp. $V^{P, \chi \Delta_P^{-1/2}}$) is nonzero.*

Corollary 2.7. *Let ψ be a character of G_n . Then $\dim \text{Hom}(\psi, \chi \times 1) = 1$ if and only if $\psi = \nu^{-p_1/2}$ and $\chi = \nu^{-n/2}$. Otherwise $\text{Hom}(\psi, \chi \times 1) = 0$.*

From highest weight theory we also obtain

Corollary 2.8. *$\chi \times 1$ has a finite-dimensional submodule if and only if either*

- (i) $\chi = \nu^{-n/2}$.
- (ii) $F = \mathbb{R}$, and $\chi = \varepsilon^k \nu^{-k-n/2}$ for some non-negative integer k .
- (iii) $F = \mathbb{C}$ and $\chi = \alpha^{(l-k)/2} \nu^{-(k+l+n)/2}$ for some non-negative integers k, l of the same parity.

2.3. Derivatives. If χ is a character of G_p with $p > 0$, we write χ' for its restriction to G_{p-1} .

We will prove the main results by induction on n , using results from [BZ77, AGS15a]. We refer the reader to those papers for the notion of *depth* for an admissible representation of G_n , and for the definition of the functor Φ which maps admissible representations of G_n of depth ≤ 2 to admissible representations of G_{n-2} . In [BZ77] this functor is denoted Φ^- .

Proposition 2.9. ([BZ77, AGS15a])

- (1) Φ is an exact functor and $\Phi(\chi_1 \times \chi_2) = \chi'_1 \times \chi'_2$.
- (2) Every subquotient of $\chi_i \times \chi_j$ has depth ≤ 2 .
- (3) If π has depth 1 then π is finite dimensional and $\Phi(\pi) = 0$.
- (4) If π has depth 2 then $\Phi(\pi) \neq 0$.

If $p_1 = 1$ (resp. $p_2 = 1$) then $\chi'_1 \times \chi'_2$ means just χ'_2 (resp. χ'_1).

2.4. Rank. The rank of representations of $GL_n(F)$ for F of zero characteristic was defined in [GS13a] using the notion of wave-front set, which is a certain union of coadjoint nilpotent orbits. The rank is defined to be the maximal among the ranks of all the matrices in the wave-front set. For archimedean F the dimension of the wave-front set equals the Gelfand-Kirillov dimension.

By Jordan's theorem the nilpotent orbits in \mathfrak{gl}_n are given by partitions of n , i.e. the sizes of Jordan blocks. By e.g. [BB89, Theorem 2] the wave-front set of $\chi_1 \times \chi_2$ is the closure of the nilpotent orbit given by the nilpotent orbit with r blocks of size 2 and $n - r$ blocks of size 1. We will shortly denote such a partition by $2^r 1^{n-r}$. Thus, $\text{rank}(\chi_1 \times \chi_2) = r$. The closure ordering on the orbits in the closure of $2^r 1^{n-r}$ is linear, and thus the wave-front set of any subquotient π of $\chi_1 \times \chi_2$ is the closure of the nilpotent orbit given by the partition $2^{\text{rank}(\pi)} 1^{n-\text{rank}(\pi)}$. The dimension of this orbit is $2\text{rank}(\pi)(n - \text{rank}(\pi))$.

Since in this paper we want to consider fields of arbitrary characteristic we give the following definition of rank

Definition 2.10. *For a subquotient π of $\chi_1 \times \chi_2$ we define the rank of π to be the number k such that $\Phi^k(\pi)$ is finite-dimensional and non-zero. In particular, if π is finite-dimensional then $\text{rank}(\pi) = 0$ and if $\pi = \chi_1 \times \chi_2$ then $\text{rank}(\pi) = \min(p_1, p_2)$.*

For F of zero characteristic this definition coincides with the one given in [GS13a]. This follows from [GS, Theorem 5.0.4] for Archimedean F and from [GS13a, Theorem B] for non-Archimedean F .

2.5. Infinitesimal characters. In this subsection we assume that F is Archimedean. We denote the Lie algebra of G by \mathfrak{g}_0 and its complexification by \mathfrak{g} , and we adopt similar terminology for subgroups of G and their Lie algebras. Let $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ be the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ and let \mathcal{Z}' be the set of its multiplicative characters. We say that $\pi \in \text{Rep}(G)$ has infinitesimal character $\xi \in \mathcal{Z}'$ if for any $z \in \mathcal{Z}$, $\pi(z)$ acts by the scalar $\xi(z)$. If π is irreducible then it has an infinitesimal character.

Lemma 2.11. *If $\pi, \pi' \in \text{Rep}(G)$ have infinitesimal characters ξ, ξ' , then $\text{Hom}_G(\pi, \pi') = 0$ unless $\xi = \xi'$.*

Fix a Cartan subalgebra and a Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the corresponding Weyl group and let ρ be the half-sum of roots of \mathfrak{h} in \mathfrak{b} .

Theorem 2.12. (Harish-Chandra) *There is an isomorphism $\mu \mapsto \xi_\mu : \mathfrak{h}^*/W \rightarrow \mathcal{Z}'$.*

We often say that a representation has infinitesimal character μ instead of ξ_μ .

Lemma 2.13. *If ϕ is a finite dimensional representation with highest weight $\mu \in \mathfrak{h}^*$ then ϕ has infinitesimal character $\mu + \rho$.*

If $P = LN$ is a parabolic subgroup, then we may choose \mathfrak{h} so that $\mathfrak{h} \subset \mathfrak{l}$; then \mathfrak{h} is a Cartan subalgebra of \mathfrak{l} as well.

Lemma 2.14. *If $\pi \in \text{Rep}(L)$ has infinitesimal character $\mu \in \mathfrak{h}^*$, then $I(P, \pi)$ has infinitesimal character μ also.*

3. FINITE-DIMENSIONAL SUBQUOTIENTS OF $\chi \times 1$

3.1. Archimedean case. Consider first the case $F = \mathbb{R}$. By the Harish-Chandra isomorphism, the characters of $\mathcal{Z}(G_n)$ can be identified with \mathbb{C}^n/S_n where S_n is the

symmetric group. Thus infinitesimal characters can be thought of as *multisets* of complex numbers.

Definition 3.1. Let $\xi = \{z_1, \dots, z_n\}$ be a multiset of complex numbers with $\operatorname{Re} z_i \geq \operatorname{Re} z_{i+1}$. We say that ξ is a *segment* if $z_i - z_{i+1} = 1$ for all i , and a *generalized segment* if $z_i - z_{i+1}$ is an integer for all i . For $s \in \mathbb{C}$, $p \in \mathbb{N}$ we denote the segment of length p with center at s by ξ_p^s .

It is well-known that the infinitesimal character of a character $\chi = \varepsilon^k \nu^s$ of $GL_p(\mathbb{R})$ is ξ_p^s , and that the infinitesimal character of an irreducible finite-dimensional representation is a generalized segment. From Lemma 2.14 we obtain that the infinitesimal character of $\chi \times 1$ is the disjoint (multiset) union $\xi_{p_1}^0 \sqcup \xi_{p_2}^s$.

For a representation of $GL_n(\mathbb{C})$ the infinitesimal character is a pair of multisets (η, ζ) , the infinitesimal character of $\chi = \alpha^k \nu^s$ is $(\xi_p^{s+k}, \xi_p^{s-k})$, and the infinitesimal character of $\chi \times 1$ is $(\xi_{p_1}^{s+k} \sqcup \xi_{p_2}^0, \xi_{p_2}^{s-k} \sqcup \xi_{p_1}^0)$.

Let us now consider the K -types of $\chi \times 1$, for $F \in \{\mathbb{R}, \mathbb{C}\}$. Here, K is the maximal compact subgroup of G_n given by $K = O(n)$ for $F = \mathbb{R}$ and $K = U(n)$ for $F = \mathbb{C}$. We denote by K^0 the connected component of K , thus $K^0 = SO(n)$ for $F = \mathbb{R}$ and $K^0 = K$ for $F = \mathbb{C}$.

Lemma 3.2. (i) If $\chi \times 1$ has the trivial K -type then χ is a power of ν .

(ii) The restriction of $\chi \times 1$ to K^0 is multiplicity free

(iii) Suppose $F = \mathbb{R}$, $\chi = \varepsilon^k \nu^s$ and let $\psi = \varepsilon^l \nu^t$ be a character of G_{p_1} , such that $\chi \times 1$ and $\psi \times 1$ have a common K -type. Then $k = l \pmod{2}$.

Proof. Let $P := P_{p_1, p_2}$, $M := K^0 \cap P$ and M^0 be the connected component of M . Consider the involution on K^0 given by conjugation by the block-diagonal matrix $\operatorname{diag}(Id_{p_1}, -Id_{p_2})$. The fixed points group of this involution is M . Thus both M and M^0 are symmetric subgroups of K^0 .

Let $\chi_K := (\chi \otimes 1)|_{P \cap K}$. By Lemma 2.4 we have $(\chi \times 1)|_K = \operatorname{Ind}_{P \cap K}^K(\chi_K)$ and $(\chi \times 1)|_{K^0} = \operatorname{Ind}_M^{K^0}(\chi_K|_M)$.

(i) follows now from Lemma 2.6 and (ii) from Lemma 2.5.

For (iii) note that if k and l have different parities then one of the representations $(\chi \times 1)|_{K^0}$ and $(\psi \times 1)|_{K^0}$ is $\operatorname{Ind}_M^{K^0}(1)$ and the other is $\operatorname{Ind}_M^{K^0}(\varepsilon \otimes 1)$. Thus, $(\chi \times 1)|_{K^0} \oplus (\psi \times 1)|_{K^0} = \operatorname{Ind}_M^{K^0}(1)$ is multiplicity free by Lemma 2.5, and thus $\chi \times 1$ and $\psi \times 1$ cannot have a common K -type. \square

Corollary 3.3. $\chi \times 1$ has at most one irreducible finite-dimensional subquotient.

Proof. Different irreducible finite-dimensional representations of the connected component G_n^0 of G_n have different infinitesimal characters. Thus only one irreducible finite-dimensional representation of G_n^0 can appear as a subquotient. It appears only once by Lemma 3.2(ii). \square

3.2. Non-archimedean case. In this subsection we assume that F is non-archimedean. Thus every irreducible finite-dimensional representation of G_n is a character.

We need the next lemma, which follows from [BZ77, Theorem 2.9].

Lemma 3.4. Let $B_n \subset G_n$ be the usual Borel subgroup, ϕ_1 and ϕ_2 be characters of B_n given by n -tuples X_1 and X_2 of characters of F^\times . Suppose that the principal series $I(B_n, \phi_i)$ have a common constituent. Then X_1 is a permutation of X_2 .

Corollary 3.5. *If $\chi \times 1$ has a one-dimensional subquotient ψ then ψ is unique and either $\chi = \nu^{-n/2}$, $\psi = \nu^{-p_2/2}$ and ψ is a submodule or $\chi = \nu^{n/2}$, $\psi = \nu^{p_1/2}$ and ψ is a quotient.*

Proof. Define a_1, a_2, a_3 to be characters of F^\times such that $\chi = a_1 \circ \det$, $\psi = a_3 \circ \det$ and $a_2 = 1$. Let $p_3 := n$ and

$$Y_i := (a_i \nu^{-(p_i-1)/2}, a_i \nu^{-(p_1-3)/2}, \dots, a_i \nu^{(p_i-1)/2}), X_1 := Y_3 \text{ and } X_2 := (Y_1, Y_2).$$

Then $\chi \hookrightarrow I(B_{p_1}, Y_1)$, $1 \hookrightarrow I(B_{p_2}, Y_2)$ and $\psi \hookrightarrow I(B_{p_3}, Y_3)$. Thus $\chi \times 1 \hookrightarrow I(B_n, X_2)$ and thus ψ is a joint subquotient of $I(B_n, X_1)$ and $I(B_n, X_2)$. Thus X_1 is a permutation of X_2 , which implies that either $a_3 = \nu^{-p_2/2}$ and $a_1 = \nu^{-n/2}$ or $a_3 = \nu^{p_2/2}$ and $a_1 = \nu^{n/2}$. In the first case we get $\chi = \nu^{-n/2}$, $\psi = \nu^{-p_2/2}$ and, by Corollary 2.7 ψ is a submodule, and in the second case $\chi = \nu^{n/2}$, $\psi = \nu^{p_1/2}$ and ψ is a quotient.

The uniqueness of ψ follows from the uniqueness of K -fixed vector (Lemma 2.4). \square

4. PROOF OF THE MAIN RESULTS

Denote $\pi := \chi \times 1$, $r := \min(p_1, p_2)$.

Proof of Theorem 1.1. We prove the statement by induction on r . For the base of the induction, note that if $r = 1$ then $\Phi(\pi)$ is 1-dimensional and thus, by Proposition 2.9, π is reducible if and only if it has either a finite-dimensional submodule or a finite-dimensional quotient. By Corollary 2.8 and Lemma 2.2, this is equivalent to the conditions of the proposition.

Note that if one of the conditions holds for χ' and $n - 2$ then it holds for χ and n . Conversely, if one of the conditions holds for χ (and n) then either it holds for χ' (and $n - 2$) or $\pi = \chi \times 1$ has a finite-dimensional submodule or quotient.

Now suppose π is irreducible but one of the conditions holds. Since we have standard intertwining operators between π and $1 \times \chi$, this implies that π is a direct summand of $1 \times \chi$. Applying Φ we get that $\Phi(\pi) = \chi' \times 1$ is a direct summand of $1 \times \chi'$. By the induction hypothesis, $\chi' \times 1$ is reducible. Applying Φ again and again if needed, we can assume (Corollary 2.8 and Lemma 2.2) that $\chi' \times 1$ has a finite-dimensional submodule or a finite-dimensional quotient. Then Corollary 2.8 and Lemma 2.2 imply that $1 \times \chi'$ cannot have a finite-dimensional constituent from the same side, contradicting having $\chi' \times 1$ as a direct summand.

Now, suppose π is reducible. If it has more than one infinite-dimensional constituent then $\chi' \times 1$ is reducible and thus, by the induction hypothesis, one of the conditions holds. Otherwise, π has either a finite-dimensional submodule or a finite-dimensional quotient. Dualizing if needed we can assume that π has a submodule and then Corollary 2.8 implies that one of the conditions holds. \square

For the proof of Theorem 1.2 we will need the following lemma.

Lemma 4.1. *If $\chi \times 1$ has a finite-dimensional irreducible subquotient ϕ , then either ϕ is a quotient and $s(\chi) > 0$ or ϕ is a submodule and $s(\chi) < 0$.*

Proof. For non-archimedean F this follows from Corollary 3.5, and thus we assume that F is archimedean and the infinitesimal character ξ is a generalized segment (if $F = \mathbb{R}$) or a pair of generalized segments (if $F = \mathbb{C}$). Assume (dualizing if needed) that $s(\chi) < 0$. If $F = \mathbb{C}$ then Theorem 1.1 and Corollary 2.8 imply that $\chi \times 1$ has a finite-dimensional submodule ϕ' . If $F = \mathbb{R}$ we let $\chi = \varepsilon^k \nu^s$ and note that there exists

a character $\psi = \varepsilon^l \nu^t$ of G_{p_1} such that ϕ can be realized as a submodule of $\psi \times 1$. This implies that $s + n/2, t + n/2$, and l have the same parity. By Lemma 3.2(iii) k also has the same parity, and thus, by Corollary 2.8, $\chi \times 1$ has a finite-dimensional submodule ϕ' . By Corollary 3.3 ϕ' must coincide with ϕ . \square

Proof of Theorem 1.2. We can assume that $s(\chi) > 0$. Then it follows from the previous lemma that π has at most one irreducible finite-dimensional subquotient ϕ , which is necessary a quotient. Now, let τ_1, \dots, τ_k be the other irreducible constituents. Then $\Phi(\tau_1), \dots, \Phi(\tau_k)$ are non-zero (not necessary irreducible) constituents of $\chi'_1 \times \chi'_2$. Arguing by induction, we can assume that $\text{rank}(\Phi(\tau_1)) > \dots > \text{rank}(\Phi(\tau_k))$ and thus $\text{rank}(\tau_1) > \dots > \text{rank}(\tau_k) > 0 = \text{rank}(\phi)$. \square

Lemma 4.2. *If F is non-Archimedean then $\chi \times 1$ has length 1 or 2.*

Proof. We prove by induction on r . The base case $r = 0$ is trivial.

If $\chi \neq \nu^{\pm n/2}$ then, by Corollary 3.5, $\chi \times 1$ has no finite-dimensional constituents. Applying the functor Φ and using Proposition 2.9 we obtain $\text{length}(\chi \times 1) \leq \text{length}(\chi' \times 1)$. The induction hypotheses implies that $\text{length}(\chi' \times 1) \leq 2$.

Now consider the case $\chi = \nu^{\pm n/2}$. Theorem 1.1 implies that $\chi' \times 1$ is irreducible and thus $\chi \times 1$ has a unique infinite-dimensional irreducible constituent. By Corollary 3.5 it also has a unique one-dimensional constituent and thus has length two. \square

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